MAC-CPTM Situations Project

Situation 29: *Trigonometric Identities*

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Prompt

While proving a trigonometric identity a student's work is this:

$$\sin x \cdot \cos x \cdot \tan x = \frac{1}{\csc^2 x}$$

$$\csc^2 x \cdot \sin x \cdot \cos x \cdot \tan x = 1$$

$$\frac{1}{\sin^2 x} \cdot \sin x \cdot \cos x \cdot \frac{\sin x}{\cos x} = 1$$

$$\cos x \cdot \frac{1}{\cos x} = 1$$

$$1 = 1$$

When asked about her reasoning, the student replies, "I just treated the equation like any algebra equation. You know, what you do to one side, you have to do to the other and then I showed it was the same as 1=1. I know 1=1 is true so the identity must be true." Is this a proper proof of the trigonometric identity?

Commentary

Proof in trigonometry is much like proof in other areas of mathematics: one cannot assume a result in order to prove it but must progress by logical steps from a given or known truth to the conclusion. When reasoning with a series of equations, it is important to show that each step necessarily and sufficiently follows from the previous step. The facts that only some manipulations of equations are reversible and that not all algebraic operations have inverses that preserve the original variable domain, can introduce logical errors in trigonometric proofs.

Focus One
It demonstrates invalid reasoning to assume a result is true when trying to prove its validity.

In the equation

$$\sin x \cdot \cos x \cdot \tan x = \frac{1}{\csc^2 x}$$

the student is asked to prove if the equality of the identities is true. By multiplying both sides by $\csc^2 x$, however, the student is acting under the assumption that the equality is indeed true. In other words, multiplying both sides of the equation presupposes that the equation is true.

This issue may be clarified by notation. For example, a question mark might be written above the equation to remind the student (and her readers) that the truth of the identity is conditional on the reversibility of the sequence of equations and the truth of the final step.

$$\sin x \cdot \cos x \cdot \tan x \stackrel{?}{=} \frac{1}{\csc^2 x}.$$

A better solution might be to use a different structure in the proof, as described in Foci Two and Three.

Focus Two
A valid argument progresses by logical steps from a given or known truth to the conclusion.

When proving trigonometric identities, it is most direct to set one side of the equation equal to itself (the known truth) and by successive manipulations (logical steps) produce the desired result (conclusion).

Applying this method to the example yields:

$$\begin{aligned}\sin x \cdot \cos x \cdot \tan x &= \sin x \cdot \cos x \cdot \tan x \\ &= \sin x \cdot \cos x \cdot \frac{\sin x}{\cos x} \\ &= \sin^2 x \\ &= \frac{1}{\csc^2 x}\end{aligned}$$

In many textbooks, students are instructed to prove trigonometric identities by manipulating just one side of the identity using substitution until it is identical to the other side. The common notation for this proof method can be misleading, as discussed in Focus 1. Although it appears that one assumes the identity to be proved, in fact, since only one side of the equation is manipulated, the list of equations can be written as a chain of equal statements with one side of the identity at each end. Thus, to prove the equality of trigonometric identities, substitutions are made on just one side of the equation while the other side is held constant.

$$\sin x \cdot \cos x \cdot \tan x = \frac{1}{\csc^2 x}$$
$$\sin x \cdot \cos x \cdot \frac{\sin x}{\cos x} = \frac{1}{\csc^2 x}$$
$$\sin x \cdot \sin x = \frac{1}{\csc^2 x}$$
$$\sin^2 x = \frac{1}{\csc^2 x}$$
$$\frac{1}{\csc^2 x} = \frac{1}{\csc^2 x}$$

Focus Three

Another valid style of argument shows that the identity to be proven is necessary and sufficient for a known identity.

In the prompt, the student shows that if the identity to be proved is true then 1 =1. In so doing, the student implies that the converse is also true. Yet a statement and its converse are not logically equivalent.

The student's proof is not valid as it stands, but might be made valid by reversing the steps. In other words, the identity is shown to be true only if one can start with a known identity (such as 1 =1) and, by working backwards, using inverse algebraic operations, one can demonstrate the equality of the identity to be proved.

$$1 = 1$$
$$\cos x \cdot \frac{1}{\cos x} = 1$$
$$\frac{1}{\sin^2 x} \cdot \sin x \cdot \cos x \cdot \frac{\sin x}{\cos x} = 1$$
$$\csc^2 x \cdot \sin x \cdot \cos x \cdot \tan x = 1$$
$$\sin x \cdot \cos x \cdot \tan x = \frac{1}{\csc^2 x}$$

Since each step follows sufficiently from the previous one, this is a valid argument that the identity to be proved is true. This procedure does not always work, however, as described in Focus 4.

Focus 4

Only some algebraic operations have inverses that preserve domain; only some manipulations of equations are reversible.

In general, algebraic manipulations of an equation show the original equation is sufficient for the new equation, but not necessary. In the following argument (Example 1), each side of an equation is squared and then the square root of each side is taken, introducing the erroneous statement, x = 3. This example demonstrates that the statement x=-3 is sufficient but not necessary for the statement $x^2 = 9$.

Example 1

$$\begin{aligned} x &= -3 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The trouble is that the inverse operation of squaring is not a function—each positive real number has two square roots. Another common problem of this type occurs when we multiply both sides of an equation by an expression that might take on the value of zero. The inverse operation would require us to divide by that expression, but this is undefined whenever the expression is zero.

Consider Example 2. Beginning with the equation x =-3, we multiply both sides of the equation by sin x. Reversing the operation, we divide by sin x, but this does not make sense whenever x is a multiple of π because sin $n\pi$ = 0, for all integers n. We do not regain x = -3, but the nonequivalent statement "x = -3 whenever x is not a multiple of π". Here, the trouble is that the inverse operation of multiplying by sin x is not defined for some of the values of x to which our original equation applied. The equation x = -3 is sufficient to show x sin x = -3 sin x but it is not necessary.

Example 2

$$\begin{aligned} x &= -3 \\ x \sin x &= -3 \sin x \\ x &= -3,\ x \neq n\pi,\ n \in Z \end{aligned}$$

The problem of the domain of an expression is particularly applicable when proving trigonometric identities. Consider the following possible identity:
$\csc x - \cos x \cdot \cot x = \sin x$. Does the following argument show the identity is true?

$$\csc x - \cos x \cdot \cot x = \sin x$$

$$\sin x\left(\frac{1}{\sin x} - \cos x \cdot \frac{\cos x}{\sin x}\right) = (\sin x)\sin x$$

$$1 - \cos^2 x = \sin^2 x$$

No, it merely shows that $\csc x - \cos x \cdot \cot x = \sin x$ is sufficient for $1 - \cos^2 x = \sin^2 x$. In fact, it is not necessary; whenever x = 0, sin x = 0 but $\csc x - \cos x \cdot \cot x$ is undefined. To see why this is the case, we try to reverse the argument. Certainly, $1 - \cos^2 x = \sin^2 x$ is true and we use the inverse operations of those used above.

$$1 - \cos^2 x = \sin^2 x$$

$$\frac{1}{\sin x}(1 - \cos x \cdot \cos x) = (\sin x)\frac{1}{\sin x};\ x \neq n\pi,\ n \in Z$$

$$\csc x - \cos x \cdot \cot x = \sin x;\ x \neq n\pi,\ n \in Z$$

Just as in the second example, dividing by sin x is undefined whenever x is a multiple of π. The order of proof used by the student in the prompt (assuming the identity to be proved and then showing it sufficiently implies a known identity) only works when each algebraic manipulation of the equation is reversible and the domain of relevant functions is preserved.

Final Commentary:

Students who are familiar with proofs by contradiction may be confused by arguments that appear to assume the conclusion (instead of its negation) in order to prove it. It may be helpful to think of this assumption as an exploration separate from the proof. The question during the phase of exploration might be, “If this equation was an identity, what other (known) identities could I confirm?” As soon as the student has found such an identity (in the prompt, the student used 1=1), the “real” proof begins. Now the question is, “Can each step be reversed?” This way of thinking highlights the fact that a proof by contradiction works backwards, starting with the negated conclusion and then contradicting a premise, whereas proofs of identities are usually direct arguments. They start with a known premise and proceed logically to the conclusion.